M.Sc. Entrance Test 2013,

Department of Statistics, Presidency University.

Answer all questions. This is closed book, closed notes test. Only a calculator is allowed during the test. Put the page number on the top of each page.

1. Let $X$ and $Y$ be identically distributed as Normal(0,1) with $Cov(X, Y) = \rho$. Show that $E(Z) = \sqrt{\frac{1-\rho}{\pi}}$, where $Z = max(X, Y)$. **(5 points)**

2. (a) If $n$ indistinguishable balls are to be placed in $n$ cells, what is the probability that exactly one cell remains empty? **(4 points)**

   (b) Let $U_1, U_2, U_3$ be independently distributed Uniform (0,1) random variables. Find the probability that the quadratic equation $U_1x^2 + 2U_2x + U_3 = 0$ will have all real roots. **(6 points)**

3. Suppose that the random variables $Y_1, Y_2, \ldots, Y_n$ satisfy $Y_i = \beta x_i + \epsilon_i$, $i = 1, 2, \ldots, n$, where $x_1, x_2, \ldots, x_n$ are fixed constants and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are iid $N(0, \sigma^2)$, $\sigma^2$ being unknown.

   (a) Find a two-dimensional sufficient statistic for $(\beta, \sigma^2)$.**(5 points)**

   (b) Find the MLE of $\beta$ and show that it is unbiased for $\beta$.**(3 points)**

   (c) Find the distribution of the MLE of $\beta$.**(4 points)**

   (d) Show that $\frac{\sum Y_i}{\sum x_i}$ is an unbiased estimator of $\beta$.**(3 points)**

   (e) Calculate the exact variance of $\frac{\sum Y_i}{\sum x_i}$. **(3 points)**

4. A non-negative random variable $U$ has cdf $F$ and density $f = F'$; its mean $\mu$ and variance $\sigma^2$ are both finite. A game is offered as follows: choose any non negative number $c$; if $U > c$ then you win the amount $c$; otherwise you win nothing.

   (a) Find an equation to characterize the value of $c$ that maximizes the expected gain. **(3 points)**

   (b) Give a characterization of $c$ in terms of the hazard rate $\lambda(x) := \frac{f(x)}{1-F(x)}$, $x > 0$. **(3 points)**

   (c) Derive $c$ explicitly for an exponential random variable with rate $\mu$. **(4 points)**

5. Give the key block arrangements of a balanced confounding for $(2^4, 2^2)$ -experiment into minimum number of replicates such that all the 2-,3- and 4-factor interactions get partially confounded. Also find the ratïo of the amount of loss in information of those interactions due to the experiment. **(7 points)**

6. Let $(X, Y)$ have a bivariate normal distribution with all parameters unknown. On the basis of a random sample of size $n$, how will you test whether the random variables $X$ and $Y$ are independently distributed? **(5 points)**

7. A unit is selected at random from a population consisting of $N$ units numbered $1, 2, \ldots, N$ and the value (say $y_1$) of a study variable $y$ is observed for the selected unit. Next every $k$-th unit is taken in a circular way and the $y$-value is observed for each of a total of $n$ units. If these values be $y_1, y_2, \ldots, y_n$, find the expectation of $T = \frac{1}{n}\sum_{i=1}^{n} y_i$, in terms of the whole set of population values. **(5 points)**

8. (a) Define a $p$-component random vector and mention the fundamental properties of its distribution function. **(6 points)**

   (b) When is a $p$-component random vector $X$ said to follows $p$-variate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$? **(2 points)**

9. (a) Determine whether $S = \{(x_1, x_2, x_3) \in R^3 : x_1 + 2x_2 + 3x_3 = 4\}$ is a subspace or not. **(2 points)**

   (b) Mention any 5 subspaces of $R^2$. **(1 point)**

   (c) Prove or disprove: If $U_1, U_2, W$ are subspaces of a vector space $V$ such that $U_1 + W = U_2 + W$, then $U_1 = U_2$. **(4 points)**

   (d) Let $V$ be finite dimensional and $U$ be a subspace of $V$ such that dimension of $U=$ dimension of $V$. Prove that $U = V$. **(4 points)**

   (e) Find the Echelon form of the following matrix and hence find the rank:
   $$M = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 2 & 5 & 0 \\ 3 & 6 & 0 & 5 \\ 4 & 8 & 1 & 2 \end{bmatrix}.$$ **(5 points)**

10. Based on a random observation $X$ drawn from Beta $(\theta, 1)$ distribution, most powerful level-$\alpha$ test for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ is given as follows:
Reject $H_0$ if $X > 1 - \alpha$ and accept otherwise.

    (a) Check whether this test is UMP against $H_1 : \theta > 1$. **(5 points)**

    (b) Examine whether the size and the level of the test are the same if $H_0$ states that $\theta < 1$. **(5 points)**

11. Suppose $X_1$ and $X_2$ are random observations drawn from a population with pdf $f_\theta(x) = \theta x^{\theta-1}$, where $\theta > 0$ and $0 < x < 1$.
Show that $log\frac{1}{\sqrt{X_1 X_2}}$ is the MVUE of $\frac{1}{\theta}$. **(6 points)**