PRESIDENCY UNIVERSITY, KOLKATA
(Admission Test 2013)
M. Sc. in Mathematics

Full Marks: 100 | Time: Two hours

Answer all the questions

1. (a) Find all the complex numbers z such that $\tan z = 2 + i$.

   (b) Show that $i \log \frac{x-i}{x+i} = \pi - 2\tan^{-1} x$ if $x > 0$. 3+2

2. (a) Give non-zero complex numbers z_1 and z_2 such that $\log z_1 z_2 \neq \log z_1 + \log z_2$.

   (b) Let $f(z)$ be a complex polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ with multiplicity $m_1, m_2, \ldots, m_r$. Find the roots of $f(z^4)$ with their multiplicity.

   (c) Let $A_1, A_2, \ldots, A_n$ be distinct elements of group $(P(X), \Delta)$, where Δ denotes the operation of symmetric difference. Find the number of elements of the subgroup $\langle A_1, A_2, \ldots, A_n \rangle$ generated by the A_i. 1+2+2

3. (a) Let $A(S)$ denote the group of all permutations of S (i.e., one-one maps of S onto itself); and let T be any non-empty proper subset of S. Let $A'(T) = \{\sigma \in A(S) : \sigma(x) = x, \forall x \in S \setminus T\}$. Show that $A'(T) \simeq A(T)$, the group of permutations of T. Also check if $A'(T)$ is a normal subgroup of $A(S)$.

   (b) Prove that the centre $Z(S_n) = \{e\}$. 3+2

4. For $N \lhd G$, and both N and G/N finitely generated, prove that G too is finitely generated.

5. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T(x, y, z) = (x + 2z, 2y + z, x + z)$ for all $(x, y, z) \in \mathbb{R}^3$. Prove that there is at least one non-zero vector $\alpha \in \mathbb{R}^3$ such that $T(\alpha) = (1 + \sqrt{2})\alpha$. Find such a vector. 4+1

6. (a) If $A = \begin{pmatrix} 1 & 2 & 2 & 1 & -1 \\ 0 & 2 & 2 & -1 & -2 \\ 2 & 6 & 2 & 1 & -4 \\ 1 & 4 & 0 & 0 & -3 \end{pmatrix}$ and $T : \mathbb{R}^5 \to \mathbb{R}^4$ be a linear transformation defined by $Tx = Ax$, $x \in \mathbb{R}^5$, find the rank and nullity of T and a basis of the null space of T.

   (b) If L, M, N are subspaces of a vector space V, give an example to show that the subspace $L \cap (M + N)$ need not be equal to $(L \cap M) + (L \cap N)$. 3+2

7. Let V be the real vector space of function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. Let $T : V \to V$ be the mapping defined by $T(f) = \frac{\partial}{\partial x} \int f(x, y)\, dy + \frac{\partial}{\partial y} \int f(x, y)\, dx$. Show that T is linear. Determine the matrix that represents T relative to the ordered basis B of V given by $B = \{x^2, xy, y^2, x, y, 1\}$. 1+4

8. Let P and Q be point of intersection of fixed curve $ax^2 + 2hxy + by^2 = 10$ and a variable straight line $lx + my = 1$. Now, if OP and OQ, O being the origin, are to remain at right angles, find the locus of the foot of the perpendicular from the origin to the line $lx + my = 1$. 5

9. The section of the cone whose guiding curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$, by the plane $x = 0$ is a rectangular hyperbola. Find the locus of the vertex of the cone. 5

10. Let A be an infinite bounded subset of $\mathbb{R}$. We know that A has at least one limit point.

    (a) Prove that A has both a smallest limit point α and a largest limit point β.

    (b) Construct an infinite set B in $\mathbb{R}$ such that $\sup B \in \{\alpha, \beta\}$. 3+2

11. (a) Prove that the sequence $\{\frac{10^{kn}}{(1.001)^n}\}$, where k is a positive real number, converges to zero. 2

(b) Let $\sum a_n$ be a series of reals, and let $\sum b_n$ be a series obtained from $\sum a_n$ by putting parantheses randomly; and let $\sum c_n$ be the series obtained from $\sum a_n$ by suppressing all zero terms. Check which of the following is valid: 3

(i) $\sum a_n$ converges $\Rightarrow \sum b_n$ converges.

(ii) $\sum b_n$ converges $\Rightarrow \sum c_n$ converges.

(iii) $\sum c_n$ converges $\Rightarrow \sum a_n$ converges.

12. Test the convergence of the sequence $\{a_n\}$ defined by $a_{n+1} = a_n \cdot \dfrac{a_n^2 + 3a}{3a_n^2 + a}$, $n \in \mathbb{N}$, where $a > 0$ and $a_1 \neq 0$. 5

13. Give an example of an ordering in $\mathbb{C}$ which makes $\mathbb{C}$ into a linearly ordered set. Check if the order make $\mathbb{C}$ into an ordered field, and if the ordering is complete. 5

14. (a) Let $D = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$; and let $f : D \to \mathbb{R}$, be any function. Show that f is a continuous function on D if and only if $\{f(\frac{1}{n})\} \to f(0)$.

(b) Let $\{a_i : i \in I\}$ be any family of non-negative reals. Define

$$\sum_{i \in I} a_i = \sup \left\{ \sum_{i \in F} a_i : F \text{ is a non-empty finite subset of } I \right\}.$$

Prove that $\sum_{i \in I} a_i < \infty$ if and only if $\{i \in I : a_i \neq 0\}$ is at most countable. 2+3

15. Find the general solution of the linear differential equation 5

$$x^2 (\log x)^2 \frac{d^2y}{dx^2} - 2x \log x \frac{dy}{dx} + [2 + \log x - 2(\log x)^2]y = x^2 (\log x)^3.$$

16. A particle of mass m moves under a central attractive force $m\mu(5r^{-3} + 8c^2 r^{-5})$ and is projected from an apse at a distance c with a velocity $\frac{3\sqrt{\mu}}{c}$. Prove that the orbit is $r = \cos \frac{2}{3}\theta$. 5

17. Three equal forces act on a body. One at the point $(1, 0, 0)$ parallel to y-axis, the second at the point $(0, 1, 0)$ parallel to z-axis and the third at the point $(0, 0, 1)$ parallel to x-axis, the axes being rectangular. Find the equation of the central axis. 5

18. The values of a polynomial $f(x)$ for different values of x are as follows:

x	-1	0	1	2	3	4
$f(x)$	-18	-8	-4	6	34	92

What more can you say about f? Also find $f(5)$. 5

19. Find a polynomial $P(x)$ of minimal degree the values of which at $x = 1, 2, 3$ and 4 are $-6, -6, -2$ and 12 respectively. Is it unique? Justify your answer. 3+2

20. (a) Let Ω be an n-element set with uniform probability and let the events $A, B \subseteq \Omega$ be independent. Show that if A has i elements, then B must have $j = k\dfrac{n}{\gcd(i,n)}$ elements, where $k \in \{0, 1, 2, ..., \gcd(i, n)\}$.

(b) Let A be an event in a probability space. Prove that the following conditions are equivalent: 2+3

i. A, B are independent events for any B.

ii. $P(A)$=0 or 1.